

Remarks on the similarity degree of an operator algebra

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Abstract

The “similarity” degree of a unital operator algebra A was defined and studied in two recent papers of ours, where in particular we showed that it coincides with the “length” of an operator algebra. This paper brings several complements: we give direct proofs (with slight improvements) of several known facts on the length which were only known via the degree, and we show that the length of a type II_1 factor with property Γ is at most 5, improving on a previous bound (≤ 44) due to E. Christensen.

MSC 2000 Classification: 46 L 07 , 46 K 99

Keywords: similarity problems, completely bounded maps.

* Supported in part by the NSF and the Texas Advanced Research Program 010366-163.

The similarity degree of a unital operator algebra A is defined in [P1] as the smallest $\alpha \geq 0$ for which there is a constant C such that any bounded morphism (= unital homomorphism) $u: A \rightarrow B(H)$ satisfies

$$(1) \quad \|u\|_{cb} \leq C\|u\|^\alpha.$$

On the other hand, an operator algebra $A \subset B(\mathcal{H})$ is said to be of length $\leq d$ if there is a constant K such that, for any n and any x in $M_n(A)$, there is an integer $N = N(n, x)$ and scalar matrices $\alpha_0 \in M_{n,N}(\mathbb{C})$, $\alpha_1 \in M_N(\mathbb{C})$, \dots , $\alpha_{d-1} \in M_N(\mathbb{C})$, $\alpha_d \in M_{N,n}(\mathbb{C})$ together with diagonal matrices D_1, \dots, D_d in $M_N(A)$ satisfying

$$(2) \quad \begin{cases} x = \alpha_0 D_1 \alpha_1 D_2 \dots D_d \alpha_d \\ \prod_{i=0}^d \|\alpha_i\| \prod_{i=1}^d \|D_i\| \leq K\|x\|. \end{cases}$$

We denote by $\ell(A)$ the smallest d for which this holds and we call it the “length” of A (so that A has length $\leq d$ is indeed the same as $\ell(A) \leq d$). It is easy to see that if $\ell(A) \leq d$ as in (2) above then for any bounded homomorphism $u: A \rightarrow B(H)$ we have $\|u\|_{cb} \leq K\|u\|^d$. In [P1], we proved the following basic result in the converse direction.

Theorem 1. *For any unital operator algebra A , we have*

$$d(A) = \ell(A).$$

Remark 2. Let $\pi: A \rightarrow B(H)$ be a completely contractive homomorphism and let $\delta: A \rightarrow B(H)$ be a π -derivation (i.e. we have $\delta(xy) = \delta(x)\pi(y) + \pi(x)\delta(y)$ for all x, y in A). Then it is easy to see that if $\ell(A) \leq d$ as in (2) above then $\|\delta\|_{cb} \leq Kd\|\delta\|$. The following result is a converse to this essentially due to Kirchberg [K], with an improvement observed in [P1].

Proposition 3. *Let A be a unital C^* -algebra. Assume there is a constant C such that for any π and δ we have $\|\delta\|_{cb} \leq C\|\delta\|$. Then $d(A) \leq C$ (hence $d(A)$ is at most equal to the integral part of C).*

There is a lot of information known on the value of $d(A)$ for various examples due to the works of E. Christensen and Uffe Haagerup ([C1-4, H]). In this note, we try to give direct and explicit factorizations for the corresponding results for $\ell(A)$. Because of Theorem 1, our results are essentially already known but they shed some light on the meaning of $\ell(A)$, and we obtain some slight improvements.

Remark 4. (i) Fix an integer n and consider x in $M_n(A)$. We denote

$$\|x\|_{(d)} = \inf \left\{ \prod_{i=0}^d \|\alpha_i\| \prod_{i=1}^d \|D_i\| \right\}$$

where the infimum runs over all possible representations $x = \alpha_0 D_1 \alpha_1 \dots D_d \alpha_d$ of the form appearing in (1) above. Clearly

$$\|x\|_{M_n(A)} \leq \|x\|_{(d)}.$$

Sometimes to avoid possible confusion in the sequel we will denote $\|x\|_{(d)}$ instead by $\|x\|_{(d,A)}$.

(ii) It is rather easy to check, say if A is unital, that $\|\cdot\|_{(d)}$ is an equivalent norm on $M_n(A)$ for each fixed n . Actually, for any x in $M_n(A)$ we have

$$\|x\|_{(1)} \leq n\|x\|,$$

but we will content ourselves with

$$(3) \quad \|x\|_{(1)} \leq n^2 \sup_{ij} \|x_{ij}\|$$

which is easy to show: we take $N = n^2$, we introduce a bijection $\sigma: [1, 2, \dots, n]^2 \rightarrow [1, \dots, N]$ and we let D in $M_N(A)$ be the diagonal matrix with entries $D_k = x_{\sigma^{-1}(k)}$.

Then we set $\alpha_0 = \sum_{i,k=1}^n e_{i\sigma(i,k)}$ and $\alpha_1 = \sum_{k,j=1}^n e_{\sigma(k,j)j}$. Clearly we have $\alpha_0 D \alpha_1 = x$, $\|\alpha_0\| \leq n$, $\|\alpha_1\| \leq n$ and $\|D\| = \sup_{ij} \|x_{ij}\|$ so that (3) follows. Note that if A has a bounded approximate unit (say on the right) (3) implies an estimate $\|x\|_{(d)} \leq C_n \sup_{ij} \|x_{ij}\|$ for some

constant C_n and $\|\cdot\|_{(d)}$ are equivalent norms on $M_n(A)$.

(iii) When A is unital (or has a contractive approximate unit, which is the case whenever A is a C^* -algebra) we have

$$\forall x \in M_n(A) \quad \|x\|_{(d+1)} \leq \|x\|_{(d)}.$$

Moreover, if A is a C^* -algebra, the results of [BP] show that

$$\forall x \in M_n(A) \quad \|x\|_{M_n(A)} = \lim_{d \rightarrow \infty} \downarrow \|x\|_{(d)}.$$

Lemma 5. *Let $A \subset B(H)$ be a closed subalgebra of $B(H)$. We assume that there are elements p, q, a_i, b_i, c_i, d_i ($1 \leq i \leq n$) in A such that*

$$a_i b_j = \delta_{ij} p, \quad c_i d_j = \delta_{ij} q, \quad \left\| \sum b_i b_i^* \right\| \leq 1, \quad \left\| \sum c_j^* c_j \right\| \leq 1.$$

Let then W_1, W_2 be two unitary $n \times n$ matrices such that

$$\forall i, j = 1, \dots, n \quad |W_1(i, j)| = |W_2(i, j)| = n^{-1/2}.$$

Then for any $x = [x_{ij}]$ in $M_n(A)$ we have the following factorization

$$[p x_{ij} q] = D_1 W_1 D_2 W_2 D_3$$

where D_1, D_2, D_3 are diagonal matrices with entries in A such that

$$\|D_1\| = \sup \|a_i\|, \quad \|D_3\| = \sup \|d_i\|, \quad \text{and} \quad \|D_2\| \leq \|x\|_{M_n(A)}.$$

Proof. Indeed, let

$$\varepsilon_{ij}^1 = n^{1/2} \overline{W_1(i, j)} \quad \text{and} \quad \varepsilon_{ij}^2 = n^{1/2} \overline{W_2(i, j)}.$$

We set $D_1(i) = a_i, D_3(i) = d_i$ and

$$D_2(k) = \sum_{ij} \varepsilon_{ik}^1 b_i x_{ij} c_j \varepsilon_{kj}^2.$$

Then it is easy to check that for any ξ, η in the unit ball of H we have

$$|\langle D_2(k)\xi, \eta \rangle| \leq \|x\| \left(\sum \|c_j \xi\|^2 \right)^{1/2} \left(\sum_i \|b_i^* \eta\|^2 \right)^{1/2} \leq \|x\|.$$

Hence $\|D_2(k)\| \leq \|x\|$. Moreover a simple calculation shows that

$$\begin{aligned} (D_1 W_1 D_2 W_2 D_3)_{ij} &= a_i \sum_k W_1(i, k) D_2(k) W_2(k, j) d_j \\ &= \sum_k W_1(i, k) [a_i D_2(k) d_j] W_2(k, j) \\ &= \sum_k W_1(i, k) [\varepsilon_{ik}^1 p x_{ij} q \varepsilon_{kj}^2] W_2(k, j) \\ &= p x_{ij} q \sum_k n |W_1(i, k)|^2 |W_2(k, j)|^2 \\ &= p x_{ij} q. \end{aligned}$$

■

Corollary 6. *Let W_1, W_2 be as in Lemma 5 and let $A = B(H)$ with $\dim H = \infty$. Then for any x in the unit ball of $M_n(A)$, there are diagonal matrices D_1, D_2, D_3 in the unit ball of $M_n(A)$ such that*

$$x = D_1 W_1 D_2 W_2 D_3.$$

More generally, the same is true for any C^ -algebra A which contains, for each n , a set of isometries s_1, \dots, s_n such that $s_i^* s_j = \delta_{ij} 1$.*

Proof. We simply take $a_i = s_i^*, b_i = s_i, c_j = s_j^*$ and $d_i = s_i$ in Lemma 5. ■

Remark 7. For any C^* -algebra, we have

$$\ell(A) \leq \ell(A^{**}).$$

Indeed, let π be the universal representation of A , and let $M = \pi(A)''$ so that $A^{**} = M$. Fix x in the unit ball of $M_n(A)$. Let $\hat{x} \in M_n(M)$ be the same element viewed in $M_n(M)$.

If $\ell(M) \leq d$, we can write $\hat{x} = \alpha_0 D_1 \alpha_1 D_2 \dots D_d \alpha_d$ with $\|D_i\| \leq 1$, D_i diagonal with entries in M , $\prod \|\alpha_i\| \leq K$. Then let (D_i^α) be a net in the unit ball of A tending in the strong operator topology (in short sot) to D_i . Then if $x^\alpha = \alpha_0 D_1^\alpha \alpha_1 \dots D_d^\alpha \alpha_d$ we have $x_{ij}^\alpha \rightarrow x_{ij}$ in sot, hence (since we are dealing with the universal representation) in the weak topology of A , hence after passing to the convex hull $x_{ij}^\alpha \rightarrow x_{ij}$ in norm, which implies by Remark 4.ii that $\|x\|_{(d)} \leq \lim_\alpha \|x^\alpha\|_{(d)} \leq K$. \blacksquare

Theorem 8. *Let A be an operator algebra. We have $\ell(A) \leq 3$ in the following cases:*

- (i) $A = B(H)$,
- (ii) A is a C^* -algebra without tracial states,
- (iii) $A = K \otimes_{\min} B$ where B is an arbitrary unital operator algebra.

Proof. (i) follows from Corollary 6. To check (ii), note that if A has no tracial states then it is well known that (for any $n \geq 1$) A^{**} contains isometries s_1, \dots, s_n such that $s_i^* s_j = \delta_{ij} = 1$. Hence by Corollary 6, $\ell(A^{**}) \leq 3$ and by Remark 7 we obtain $\ell(A) \leq 3$. To check (iii), note that by (3) it suffices to be able to factorize all elements x of a dense subset of the unit ball of $M_n(A)$. Hence we may assume that the entries x_{ij} of x lie in a dense linear subspace $V \subset A$. We will use the linear subspace $V \subset K \otimes_{\min} B$ spanned by $[e_{ij} \otimes b]$. Then consider x in the unit ball of $M_n(V)$, so that there is an integer m such that if p, q in V are defined as $p = q = \sum_{i=1}^m e_{ii} \otimes 1_B$, we have $x_{ij} = p x_{ij} q$. Thus, if we set $a_i = p s_i^* \otimes 1_B$, $d_i = s_i p \otimes 1_B$, $b_i = s_i \otimes 1_B$ and $c_i = s_i^* \otimes 1_B$, we obtain $\|x\|_{(3)} \leq \|x\|_{M_n(A)}$ when $A = K \otimes_{\min} B$ and we conclude that $\ell(K \otimes_{\min} B) \leq 3$. \blacksquare

Theorem 9. *Let $A \subset B(H)$ be a C^* -subalgebra, generating a von Neumann algebra M . Assume that M is a II_1 -factor. Then $\ell(M) \leq \max\{\ell(A), 3\}$.*

Corollary 10. Let R be the hyperfinite II_1 -factor. Then $\ell(R) \leq 3$.

The last two statements are easy consequences of the following.

Lemma 11. *Let $x = (x_{ij})$ in the unit ball of $M_n(M)$ be such that $\sum_{ij} \|x_{ij}\|_2^2 < \varepsilon^2$, then there are projections p, q in M with $\tau(p) \leq \frac{1}{n}$, $\tau(q) \leq \frac{1}{n}$ such that*

$$x_{ij} = p x_{ij} q + y_{ij}$$

with $\|y_{ij}\| \leq 2\varepsilon\sqrt{n}$.

Proof. Consider $a = \left(\sum_{ij} x_{ij}^* x_{ij} \right)^{1/2}$. Then if E_a the spectral measure of a , we let $q = E_a[\varepsilon\sqrt{n}, \infty]$ so that $\tau(q) \leq \frac{1}{n}$

$$x_{ij} = x_{ij} q + x_{ij} (1 - q)$$

so that

$$\begin{aligned}\|x_{ij}(1-q)\|^2 &= \|(1-q)x_{ij}^*x_{ij}(1-q)\| \\ &\leq \|(1-q)a^2(1-q)\| \\ &\leq \varepsilon^2 n.\end{aligned}$$

Hence $\|x_{ij}(1-q)\| \leq \varepsilon\sqrt{n}$. Now let $b = \left(\sum_{ij} x_{ij}x_{ij}^*\right)^{1/2}$ and $p = E_b[\varepsilon\sqrt{n}, \infty]$. We have

$$x_{ij}q = px_{ij}q + (1-p)x_{ij}q$$

and again $\tau(p) \leq \frac{1}{n}$ but

$$\|(1-p)x_{ij}q\| \leq \|(1-p)x_{ij}\| \leq \varepsilon\sqrt{n}.$$

■

Remark. As a consequence of Lemma 11, if $C(d)$ denotes the unit ball of the norm $\|\cdot\|_{(d)}$ on $M_n(M)$ (see Remark 4) and if $d \geq 3$ we have:

$$\overline{C(d)}^{L_2(M_n(M))} \subset 3C(d).$$

Proof of Theorem 9. By the Kaplansky density theorem, for any $\varepsilon > 0$, any z in the unit ball of $M_n(M)$ can be written as $z = z' + x$ with z' in the unit ball of $M_n(A)$ and $\sum \|x_{ij}\|_2^2 < \varepsilon^2$. With the notation of Remark 5, if $d = \ell(A)$ we have $\|z'\|_{(d)} \leq K$ for some fixed constant K (independent of n). Thus we are reduced to the factorization of x . But with the notation of Lemma 11 we have $x_{ij} = px_{ij}q + y_{ij}$ and Lemma 4 ensures that $\|(px_{ij}q)\|_{(3)} \leq 2$. Thus we are reduced to estimate $y = (y_{ij})$, but by (3) we have $\|y\|_{(1)} \leq 2\varepsilon n^{5/2}$, hence we finally conclude that if $d \geq 3$ we have

$$\|z\|_{(d)} \leq K + 2 + 3\varepsilon n^{5/2}$$

and if $d < 3$ we have the same majorization for $\|z\|_{(3)}$. Thus we obtain Theorem 9. ■

Remark 12. It is proved in [P1] that $d(A) \leq 2$ implies that A is “semi-nuclear” in the following sense: for any *-representation $\pi : A \rightarrow B(H)$, the generated von Neumann algebra $M = \pi(A)''$ is injective whenever it is semi-finite. Note that nuclear implies semi-nuclear. By well known results, if G is any discrete group, and if either $A = C^*(G)$ or $A = C_\lambda^*(G)$, then A is semi-nuclear iff G is amenable, or equivalently iff A is nuclear. In general it seems to be open whether conversely semi-nuclear implies nuclear. However, the results of [A] imply that $B(H)$ is not semi-nuclear (here $\dim(H) = \infty$). As pointed out to me by Narutaka Ozawa, it is easy to adapt the argument in [A] to show that the hyperfinite II_1 factor R is not semi-nuclear, and actually that no II_1 factor can be semi-nuclear. Thus we have

$$\ell(M) \geq 3$$

for any II_1 factor M .

Let M be a II_1 -factor with (Murray and von Neumann's) property Γ . This means that there is a net of unitaries (u_α) in M with zero trace which are asymptotically central, i.e. are such that $\|u_\alpha t - t u_\alpha\|_2 \rightarrow 0$ for any t in M .

By a result of Dixmier [D], we can then find “many” asymptotically central elements, in particular for any n there exists a net $(p_1^\alpha, \dots, p_n^\alpha)$ of orthogonal decompositions of the identity in M with $\tau(p_i^\alpha) = \frac{1}{n}$ for all i, α and such that $(p_i^\alpha)_\alpha$ is asymptotically central for each $i = 1, \dots, n$. In particular, we have for any t in M , $\lim_\alpha \|t - \sum_1^n p_i^\alpha t p_i^\alpha\|_2 = 0$ (indeed note that $\|t - [ptp + (1-p)t(1-p)]\|_2 = \|t - [ptp + t - pt - tp + ptp]\|_2 = \|p(tp-pt) + (pt-tp)p\|_2$.) Now fix x in the unit ball of $M_n(M)$ and let $x_{ij}^\alpha = \sum_{m=1}^n p_m^\alpha x_{ij} p_m^\alpha$. We have $\|x^\alpha\|_{M_n(M)} \leq \|x\|_{M_n(M)}$ and

$$\lim_{\alpha \rightarrow \infty} \|x - x^\alpha\|_{L^2(M_n(M))} = 0.$$

This allows us to improve the main result of [C4], as follows (the estimate given in [C4] is $d(M) \leq 44$).

Theorem 13. *If M is II_1 -factor with property Γ then $\ell(M) \leq 5$.*

Proof. By the preceding remarks and by Lemma 11, it suffices to show that if p_1, \dots, p_n are disjoint projections in M with $\tau(p_i) = \frac{1}{n}$ we have

$$\left\| \left[\sum_{m=1}^n p_m x_{ij} p_m \right]_{ij} \right\|_{(5)} \leq \|x\|_{M_n(M)}.$$

This is an immediate consequence of the next lemma, where we use freely the notation introduced in Remark 4.

Lemma 14. *Let X_1, \dots, X_n be in $M_n(M)$ with $\|X_m\|_{(d)} \leq 1$ for any $m = 1, \dots, n$. Then if we let*

$$y_{ij} = \sum_{m=1}^n p_m X_m(i, j) p_m$$

with p_m as above, we have

$$\|y\|_{(d+2)} \leq 1.$$

Proof. We have

$$y = \alpha X \alpha^*$$

where $\alpha \in M_{n,n^2}(\mathbb{C})$ and $X \in M_{n^2}(M)$ are defined as:

$$\alpha = \sum \bar{e}_{1m} \otimes 1 \otimes p_m$$

and $X = \sum e_{mm} \otimes X_m$. Here \bar{e}_{1m} denotes the canonical basis of $M_{1,n}(\mathbb{C})$ and we use the usual identifications

$$M_{1,n} \otimes M_n = M_{n,n^2} \quad \text{and} \quad M_n \otimes M_n = M_{n^2}.$$

It is easy to see, that in $M_{n^2}(M)$ we have $\|X\|_{(d)} \leq \sup_m \|X_m\|_{(d)}$. Thus it suffices to show that $\|\alpha\|_{(1)} \leq 1$, since $\|\alpha^*\|_{(1)} \leq 1$ follows by transposition. But the latter follows from the following identity

$$\alpha = \sum_1^n \bar{e}_{1m} \otimes 1 \otimes p_m = \alpha_0 DW$$

where $\alpha_0 = \frac{1}{\sqrt{n}} \sum_1^n \bar{e}_{1m} \otimes 1 \otimes 1$

$$D = \sum_1^n e_{mm} \otimes 1 \otimes \sum_{i=1}^n \bar{W}_{im} \sqrt{n} p_i$$

and

$$W = \sum_{ij} W_{ij} e_{ij} \otimes 1 \otimes 1.$$

Indeed, we have

$$\begin{aligned} \alpha_0 DW &= \frac{1}{\sqrt{n}} \sum_{m=1}^n \sum_{j=1}^n \bar{e}_{1j} \otimes 1 \otimes \sum_{i=1}^n \sqrt{n} \bar{W}_{im} W_{mj} p_i \\ &= \sum_j \bar{e}_{1j} \otimes 1 \otimes \sum_i (W^* W)_{ij} p_i \\ &= \sum_{j=1}^n \bar{e}_{1j} \otimes 1 \otimes p_j = \alpha. \end{aligned}$$

Whence we obtain

$$\|\alpha\|_{(1)} \leq \|\alpha_0\| \|D\| \|W\| \leq 1.$$

■

We will now discuss the following which goes back to 1955.

Kadison's conjecture ([K]). Any C^* -algebra has the following similarity property (SP). Every bounded homomorphism $u: A \rightarrow B(H)$ is similar to a $*$ -homomorphism, i.e. there is an invertible $\xi: H \rightarrow H$ such that the homomorphism u_ξ defined by $u_\xi(x) = \xi^{-1}u(x)\xi$ ($\forall x \in A$) is a $*$ -homomorphism, which means that $u_\xi(x^*) = u_\xi(x)^*$ ($\forall x \in A$).

By a result due to Haagerup [H], it is known that (if u is unital) u is similar to a $*$ -homomorphism iff u is c.b. Moreover, we have

$$\|u\|_{cb} = \inf\{\|\xi^{-1}\| \|\xi\|\}$$

where the infimum runs over all ξ for which u_ξ is a $*$ -homomorphism (or equivalently for which $\|u_\xi\| = 1$).

By the results of [P1], we have

Proposition 15. *If Kadison's conjecture is true then there is a fixed d_0 such that any C^* -algebra has length $\leq d_0$.*

Proof. By [P1] a C^* -algebra A satisfies (SP) iff $\ell(A) < \infty$. Assume that there are C^* -algebras A_n such that $\ell(A_n) \rightarrow \infty$ when $n \rightarrow \infty$. Then let $A = \bigoplus_n A_n$ be (say) the C^* -algebra formed of sequences $x = (x_n)_n$ with $x_n \in A_n$ such that $x_n \rightarrow 0$ when $n \rightarrow \infty$. Clearly $\ell(A) \geq \ell(A_n)$ for all n hence $\ell(A) = \infty$. Thus if Kadison's conjecture is correct, $\ell(A) < \infty$ for any A , whence Proposition 15 follows. \blacksquare

Let A be an operator algebra. For any set I , we denote by $\ell_\infty(I, A)$ the operator algebra formed of all bounded families $(x_i)_{i \in I}$ with $x_i \in A$ for all i in I , equipped with the norm $\|x\| = \sup_{i \in I} \|x_i\|$. In the next result, we show that the length of $\ell_\infty(I, A)$ (with I infinite) is closely related to the possibility of obtaining the factorization described in (2) above, with the size N and the scalar factors depending only on n and not of $x \in M_n(A)$.

Proposition 16. *Let A be an operator algebra. For any set I , we denote by $\ell_\infty(I, A)$ the operator algebra formed of all bounded families $(x_i)_{i \in I}$ with $x_i \in A$ for all i in I , equipped with the norm $\|x\| = \sup_{i \in I} \|x_i\|$. Fix an integer $d \geq 1$. The following assertions*

are equivalent.

- (i) *For any set I , $\ell(\ell_\infty(I, A)) \leq d$.*
- (ii) *For any countable set I , $\ell(\ell_\infty(I, A)) \leq d$.*
- (iii) *There is a constant K such that for any n there is an integer $N = N(n)$ and scalar matrices of norm 1*

$$\alpha_0 \in M_{n,N}(\mathbb{C}), \alpha_1 \in M_N(\mathbb{C}), \dots, \alpha_{d-1} \in M_N(\mathbb{C}), \alpha_d \in M_{N,n}(\mathbb{C})$$

such that for any x in $M_n(A)$ there are diagonal matrices D_1, \dots, D_d in $M_N(A)$ with

$$\prod_{i=1}^d \|D_i\| \leq K\|x\| \text{ and satisfying}$$

$$x = \alpha_0 D_1 \alpha_1 D_2 \dots D_d \alpha_d.$$

Proof. Note that we have canonically

$$M_n(\ell_\infty(I, A)) = \ell_\infty(I, M_n(A)).$$

Assume (iii). Consider x in $M_n(\ell_\infty(I, A))$. Let $(x(i))_{i \in I}$ be the associated element in $\ell_\infty(I, M_n(A))$ with

$$\|x\| = \sup_{i \in I} \|x(i)\|.$$

By (iii) we can find for each i in I diagonal elements $D_1(i), \dots, D_d(i)$ in $M_N(A)$ such that $x(i) = \alpha_0 D_1(i) \dots D_d(i) \alpha_d$. Let D_1, \dots, D_d be the corresponding (diagonal) elements of $M_N(\ell_\infty(I, A))$. Then we clearly have $x = \alpha_0 D_1 \alpha_1 \dots D_d \alpha_d$ and (2) holds. So we obtain (i).

Conversely, assume (i). Let I be the unit ball of $M_n(A)$. Let $x: I \rightarrow M_n(A)$ ($i \rightarrow x(i)$) be the inclusion mapping. Clearly $x \in \ell_\infty(I, M_n(A))$ with $\|x\| = 1$. We can also view x as an element of $M_n(\ell_\infty(I, A))$. Then, if (i) holds we can find $\alpha_0, \dots, \alpha_d$ with norm 1 and D_1, \dots, D_d diagonal in $M_N(\ell_\infty(I, A))$ such that $x = \alpha_0 D_1 \dots D_d \alpha_d$ and $\Pi \|D_i\| \leq K \|x\| \leq K$. (Here $N = N(n, x)$ but since x is a fixed canonical element, actually N depends only on n .) Taking the i -th coordinate, we obtain

$$\forall i \in I \quad x(i) = \alpha_0 D_0(i) \alpha_1 \dots D_d(i) \alpha_d.$$

Hence we conclude (by homogeneity) that (iii) holds.

This shows that (i) \Leftrightarrow (iii). Since (i) \Rightarrow (ii) is trivial, it remains only to show that (ii) \Rightarrow (iii). Assume (ii). Fix an integer n . Then we may observe that $\ell_\infty(\mathbb{N}, A)$ is of length $\leq d$ with K (independently of n or x) and $N(n, x)$ as defined before (2) but moreover with matrices $\alpha_0, \alpha_1, \dots, \alpha_d$ with *rational* coefficients. Indeed, a simple density argument (it is convenient here to invoke (3)) establishes this fact (perhaps at the cost of a K slightly worse than the original one).

Then let S_N be the set of all $(d+1)$ -tuples $(\alpha_0, \dots, \alpha_d)$ with rational coefficients in $M_{n,N} \times M_{N,N} \times \dots \times M_{N,n}$ with $\|\alpha_0\|, \dots, \|\alpha_d\| \leq 1$, and let

$$I = \bigcup_{N \geq 1} \{N\} \times S_N.$$

Clearly, I is countable.

We claim that there is a p in I say $p = (N, (\alpha_0, \dots, \alpha_d))$ with $(\alpha_0, \dots, \alpha_d) \in S_N$ such that (iii) holds relative to N and $(\alpha_0, \dots, \alpha_d)$ (i.e. the same N and the same $(\alpha_0, \dots, \alpha_d)$ work for any x in the unit ball of $M_n(A)$).

Indeed, if we assume otherwise. Then for any p in I , there is x_p in the unit ball of $M_n(A)$ such that whenever we have an equality

$$x_p = \alpha_0 D_1 \alpha_1 \dots D_d \alpha_d$$

with D_1, \dots, D_d as in (iii) then we must have

$$(4) \quad \prod_{j=1}^d \|D_j\| > K.$$

Let $x = (x_p)_{p \in I}$. Note that x is in the unit ball of $\ell_\infty(I; M_n(A)) = M_n(\ell_\infty(I, A))$. Applying our original observation about *rational* coefficients, we find that there is an N and $q = (\alpha_0, \dots, \alpha_d)$ in S_N such that x can be written as $x = \alpha_0 D_1 \dots D_d \alpha_d$ with D_1, \dots, D_d diagonal in $M_N(\ell_\infty(I, A))$ such that $\prod \|D_j\| \leq K$. In particular, if we restrict this equality to the p -th coordinate of x with $p = (N, q)$ we find a factorization of the form $x_p = \alpha_0 D_1(p) \dots D_d(p) \alpha_d$ which contradicts (4).

This proves the above claim, and thus concludes the proof that (ii) \Rightarrow (iii). \blacksquare

Taking into account Proposition 16, a close look at the proof of Theorem 9 immediately yields:

Corollary 17. *Let M be a II_1 -factor with property Γ . Then we have $\ell(\ell_\infty(I, M)) \leq 5$ for any set I .*

Let $B \subset A$ be an inclusion between operator algebras. In order to compare the norms $\|\cdot\|_{(d)}$ defined above in Remark 4 for A and for B , we denote the respective norms by $\|\cdot\|_{(d,A)}$ and $\|\cdot\|_{(d,B)}$.

Sublemma 18. *Let B be an operator algebra and let $A = M_n(B) \approx B \otimes M_n$. Fix r, s with $1 \leq r, s \leq n$. Let $j_{r,s}: B \rightarrow A$ be the (“inclusion”) mapping defined by $j_{r,s}(b) = b \otimes e_{rs}$ ($b \in B$). Then, for any x in $M_n(B)$ we have (r, s remaining fixed)*

$$\|[j_{r,s}(x_{ij})]\|_{(3,A)} \leq \|x\|_{M_n(B)}.$$

Proof. We apply Lemma 5 with $q = p = 1 \otimes e_{rs}$, $a_i = 1 \otimes e_{ri}$, $b_j = 1 \otimes e_{js}$, $c_i = 1 \otimes e_{ri}$, $d_j = 1 \otimes e_{js}$. \blacksquare

Sublemma 19. *Again let $A = M_n(B) \approx B \otimes M_n$ as before and let $j: B \rightarrow A$ be the mapping defined by $J(b) = b \otimes 1$. Then for any $x = (x_{ij})$ in $M_n(B)$ we have*

$$\|[J(x_{ij})]\|_{(5,A)} \leq \|x\|_{M_n(B)}.$$

Proof. This is the same argument as for Lemma 14. We can write

$$J(x_{ij}) = x_{ij} \otimes 1 = \sum_{m=1}^n p_m X_m(i, j) p_m$$

where $p_m = 1 \otimes e_{mm} \in A$ and

$$X_m(i, j) = x_{ij} \otimes e_{mm} = J_{mm}(x_{ij}).$$

Then arguing as for Lemma 14, we obtain Sublemma 18. \blacksquare

Remark 20. Note that the factors W_1, W_2, W (and their sizes) which appear when we spell out explicitly the factorizations corresponding to Sublemmas 18 and 19 depend only on n and not on x .

Hence we obtain:

Proposition 21. *Let C be the CAR algebra $C = (M_2)^{\otimes \mathbb{N}}$ or any infinite C^* -tensor product of matrix algebras. Then for any set I , we have $\ell(\ell_\infty(I, C)) \leq 5$.*

Proof. Consider x in the open unit ball of $M_n(C)$. By density we may assume that all entries x_{ij} belong to $C_N \otimes 1 \simeq C_N$ where $C_N = M_2 \otimes \cdots \otimes M_2$ (N -times). Now assume without loss of generality that $n = 2^k$. Note that the inclusion $C_N \rightarrow C$ can be factored as $C_N \xrightarrow{J} C_N \otimes M_{2^k} \xrightarrow{\pi} C$ where $J(b) = b \otimes 1$ as above. Thus since π is a $*$ -homomorphism we have $\|x\|_{(5, C)} \leq \|x\|_{M_n(C)}$. By Remark 20 and Proposition 16, we conclude that $\ell(\ell_\infty(I, C)) \leq 5$. ■

The II_1 factor associated with the free group with at least two generators is a typical example of one failing property Γ , and it might be a counterexample to Kadison's conjecture. But actually, we feel that the following should be true.

Conjecture. Let M be the von Neumann algebra formed of all norm-bounded sequences (x_n) with $x_n \in M_n$ for each n , equipped with the sup-norm and let $N = \ell_\infty(\mathbb{N}, M)$. Then N (and perhaps even M) is a counterexample to Kadison's conjecture. In other words, its "length" is infinite.

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